The algebra of the Lax connection for T-dual models

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Abstract

We study the relation between T-duality and integrability. We develop the Hamiltonian formalism for a principal chiral model on general group manifold and on its T-dual image. We calculate the Poisson bracket of Lax connections in the T-dual model and show that they are non-local, which is opposite to the Poisson brackets of the Lax connection in the original model. We demonstrate these calculations on two specific examples: sigma model on S^2 and sigma model on AdS_2.

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1. Introduction and summary

One of the most remarkable achievements in string theory in the past few years was the discovery of the integrability of $N = 4$ superconformal Yang–Mills (SYM) theory in its planar limit together with the integrability of $AdS_5 \times S^5$ superstring [1–3][1]. Indeed, the well-known classical integrability of bosonic string on $AdS_5 \times S^5$ was extended to the κ-symmetric Green–Schwarz superstring [11, 12] or to the pure spinor formulation of superstring as well [16–30][2]. For example, it was found that the classical superstring possesses an infinite number of conserved non-local charges. These charges have their counterpart in planar gauge theory at weak coupling in the spin-chain formulation for the dilatation operator [13, 14].

It is also believed that the integrability of the $N = 4$ SYM should have an impact on the spectrum of other observables in the theory, for example on the structure of the expectation values of certain Wilson loops. The dual formulation of these objects in the AdS/CFT correspondence is the partition functions of *open* $AdS_5 \times S^5$ strings that end on some contours at the boundary of AdS_5 [35, 36]. Moreover, it turned out that the open string description of Wilson loops is directly related to the T-duality in $AdS_5 \times S^5$ [37, 38][3], where T-dual

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[1] For review and extensive list of references, see [4–9].
[2] For review, see [31–34].
[3] For review and extensive list of references, see [39–41].

formulation appears to be important in the discovery of a connection between maximally helicity-violating (MHV) gluon scattering amplitudes and special Wilson loops (defined on contours formed by light-like gluon momentum vectors). The classical $SO(4, 2)$ conformal symmetry of the T-dual AdS$_5$ geometry seems to be related to a mysterious 'dual' conformal symmetry that was observed in the momentum-space integrands of loop integrals for planar gluon scattering amplitudes [42–45]. From the string theory point of view, this dual conformal symmetry of the AdS$_5 \times S^5$ sigma model could be related to the presence of hidden symmetries in T-dual string theory. In fact, since T-duality is an on-shell symmetry or, in other words, it maps classical solutions to the classical one and since the T-dual geometry is again AdS$_5 \times S^5$, we can expect that the T-dual model is also integrable and consequently possesses an infinite number of conserved charges that should correspond to generators of some symmetries of dual Wilson lines in $N = 4$ SYM.

In the phase-space formulation of string theory, the statement that T-duality is on-shell symmetry is that T-duality is canonical transformation [63, 64]. This fact has a significant consequence on the calculation of the Poisson brackets of Lax connection in T-dual theory. In fact, how the Lax connection in original integrable theory is mapped to its T-dual counterpart is an important question[4]. These problems were recently discussed in papers [46–48]. In particular, the paper [46] discussed the interplay between T-duality and integrability on two examples: the two-sphere S^2 and the two-dimensional anti-de Sitter space AdS$_2$. It was argued there that in order to perform T-duality explicitly, we have to express T-dual coordinates in terms of the original ones. Further, the fact that this relation is non-local is also important, and hence it is a non-trivial task to find the T-duality image of Lax connection since Lax connection—opposite to the sigma model action—depends explicitly on the coordinate that parameterizes the T-duality direction. On the other hand, it was shown in [46] that it is possible to eliminate the explicit dependence of the Lax connection on the T-duality direction coordinate with the help of some special field redefinition that preserves the flatness of the Lax connection of the original theory. The new Lax connection depends on the derivatives of the isometric coordinate only. Then T-duality on these flat currents can be easily implemented and it is easy to find the T-dual flat currents.

The goal of this paper is to further study the properties of the Lax connection in the T-dual background. We are mainly interested in the calculation of the Poisson bracket of the T-dual Lax connections. We find an explicit form of these Poisson brackets and argue that they have exactly the same form as the brackets introduced in [59, 60]. Then, by following the discussion presented in these papers and reviewed in the appendix, we can argue that T-dual theory possesses an infinite number of conserved charges that are in involution[5] in the sense that their Poisson brackets commute. On the other hand, it is desirable to find an explicit form of matrices r, s that appear in these Poisson brackets and are crucial for the study of classical or quantum mechanical integrability of given theory [59, 60]. We show that even though it is straightforward to find the forms of these matrices in the case of the principal chiral model, it is very difficult to find their forms in the T-dual model. In fact, we will argue that the original constant matrices r, s map under T-duality to non-local expressions. We claim that this is in accord with the observation that local charges map to non-local ones under T-duality.

The organization of this paper is as follows. In section 2, we review the calculation of the Poisson bracket of the Lax connection in the case of a S^2 sigma model and show that in this case the matrices r, s are constant. Then in section 3, we perform the calculation of the Poisson bracket of the Lax connection for the sigma model that is T-dual to the sigma model

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[4] For some earlier works that discuss this problems, see [50–58].

[5] Of course, up the subtlety of the definition of the Poisson bracket of monodromy matrices in the case when the initial and final points coincide. For a discussion of these problems, see [59–62].

on S^2. We review the construction of the Lax connection performed in [46] and express it as a function of canonical variables. Then we calculate the Poisson brackets of Lax connection for T-dual action and argue for the integrability of given theory. In section 4, we study another example of the T-dual sigma model that arises by performing T-duality along the non-compact direction of AdS$_2$. We again calculate the Poisson brackets of Lax connection that is an image of the original Lax connection under T-duality. We however stress that T-duality in the case of the AdS$_2$ background is special since now T-duality along the non-compact direction leads to a background that is again AdS$_2$. It is then natural to presume that this background possesses Lax connection that has the same form as the original one, now expressed as a function of T-dual variables. One can then expect that these two Lax connections are related, and it was shown, for example, in [47] this is really true. We plan to discuss the Hamiltonian formulation of these Lax connections and relations between them in a forthcoming publication. Finally, in section 5, we present a general analysis of the calculation of the Poisson bracket between Lax connections in the sigma model that is related to the original one by duality that is canonical transformation in phase space. We define the Lax connection in a dual model in two steps. We first perform the gauge transformation that preserves the flatness of a given Lax connection. In the second step, we express this Lax connection as a function of phase space variables that are related to the original ones by canonical transformations. In fact, it is well known that T-duality can be considered as some kind of canonical transformations. We find that the Poisson brackets of dual Lax connections are non-local; however, their forms again imply that the new theory possesses an infinite number of conserved charges that are in involution.

Let us outline our result. We derive Poisson brackets of Lax connections in sigma models that are T-dual to original integrable models. We argue that the new integrable models contain an infinite number of conserved charges that are in involutions even if the Poisson brackets of Lax connections are non-local and the matrices r, s are functions of phase space variables. We hope that these calculations can be useful for a further study of relations between T-duality and integrability.

2. Review of the S^2 sigma model and Poisson bracket of its Lax connection

In this section, we give a brief review of the calculation of the Poisson bracket of the Lax connection of the sigma model on S^2. The main goal is to demonstrate the difference between the straightforward calculations given here with respect to the analysis of the Poisson bracket of the Lax connection in the case of the T-dual model that will be presented in the following section.

We start with the action that governs the dynamics of the string on S^2:

$$S = -\frac{1}{2} \int d^2\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} [\partial_\alpha \Theta \partial_\beta \Theta + \sin^2 \Theta \partial_\alpha \Phi \partial_\beta \Phi], \tag{2.1}$$

where $\gamma_{\alpha\beta}, \alpha, \beta = 0, 1$, is the worldsheet metric and $\sigma^\alpha, \sigma^0 = \tau$ and $\sigma^1 = \sigma$ are the worldsheet coordinates.

We start with the observation that this theory possesses Noether currents in the form

$$j^1 = -\frac{1}{\sqrt{2}} [\sin(\Phi) d\Theta + \sin \Theta \cos \Theta \cos \Phi \, d\Phi],$$
$$j^2 = -\frac{1}{\sqrt{2}} [\cos \Phi \, d\Theta - \sin \Theta \cos \Theta \sin \Phi \, d\Phi], \tag{2.2}$$
$$j^3 = -\frac{1}{\sqrt{2}} \sin^2 \Theta \, d\Phi,$$

where $df \equiv \partial_\alpha f \, d\sigma^\alpha$. Then, it is easy to see that

$$\gamma^{\alpha\beta} j_\alpha^A j_\beta^B K_{AB} = -\frac{1}{2} \gamma^{\alpha\beta} [\partial_\alpha \Theta \partial_\beta \Theta + \sin^2 \Theta \partial_\alpha \Phi \partial_\beta \Phi], \tag{2.3}$$

where $A, B, \ldots = 1, 2, 3$ and the Cartan–Killing form is given by

$$K_{AB} = \text{Tr}(T_A T_B) = \text{diag}(-1, -1, -1). \tag{2.4}$$

Then using this result, we can rewrite action (2.1) in the form

$$S = \int \mathrm{d}^2\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \, \text{Tr} \, j_\alpha j_\beta \tag{2.5}$$

that clearly demonstrates the fact that the dynamics of string on S^2 is governed by the principal model action. Further, we observe that currents (2.3) are flat:

$$\partial_\alpha j_\beta^A - \partial_\beta j_\alpha^A + j_\alpha^B j_\beta^C f_{BC}{}^A = 0, \tag{2.6}$$

where the structure constants are defined as

$$f_{BC}{}^A = -\frac{1}{\sqrt{2}} \epsilon_{BCD} K^{DA}. \tag{2.7}$$

Here, ϵ_{ABC} is totally antisymmetric with $\epsilon_{123} = -1$. In what follows, we will be more general and introduce the general coordinates x^M on manifold M that in the particular case of $M = S^2$ are $x^M = (\Theta, \Phi)$. As the next step, we introduce E_M^A in order to write the current j^A in the form

$$j_\alpha^A = E_M^A \partial_\alpha x^M. \tag{2.8}$$

Then the conjugate momenta p_M defined as $\frac{\delta S}{\delta \partial_\tau x^M}$ take the form

$$p_M = \sqrt{-\gamma} \gamma^{\tau\alpha} K_{AB} E_M^A E_N^B \partial_\alpha x^N. \tag{2.9}$$

Further, we define the current j_P^A as

$$j_P^A = -\sqrt{-\gamma} \gamma^{\tau\alpha} E_M^A \partial_\alpha x^M = -K^{AB} E_B^M p_M, \tag{2.10}$$

where E_B^M is inverse of E_M^A. Then it is easy to calculate the Poisson bracket

$$\begin{aligned}
\left\{ j_\sigma^A(\sigma), j_P^B(\sigma') \right\} &= -E_N^A(\sigma) E_C^N(\sigma') K^{BC} \partial_\sigma \delta(\sigma - \sigma') - \partial_N E_M^A \partial_\sigma x^M K^{BC} E_C^A \delta(\sigma - \sigma') \\
&= -K^{AB} \partial_\sigma \delta(\sigma - \sigma') - j_\sigma^D f_{DC}{}^A K^{CB} \delta(\sigma - \sigma'),
\end{aligned} \tag{2.11}$$

using

$$\begin{aligned}
\partial_{\sigma'} \delta(\sigma - \sigma') &= -\partial_\sigma \delta(\sigma - \sigma'), \\
f(\sigma') \partial_\sigma \delta(\sigma - \sigma') &= f(\sigma) \partial_\sigma \delta(\sigma - \sigma') + \partial_\sigma f(\sigma) \delta(\sigma - \sigma')
\end{aligned} \tag{2.12}$$

and also the relations

$$\partial_N E_M^A - \partial_M E_N^A + E_M^B E_N^C f_{BC}{}^A = 0, \qquad \partial_P E_C^N = -E_C^M \partial_P E_M^D E_D^N \tag{2.13}$$

that follow from the fact that the current $j^A = E_M^A \mathrm{d}x^M$ is flat. In the same way, we obtain

$$\left\{ j_P^A(\sigma), j_P^B(\sigma') \right\} = -j_P^C(\sigma) f_{CD}{}^A K^{DB} \delta(\sigma - \sigma'). \tag{2.14}$$

Now we are ready to determine the Poisson brackets of Lax connection for the S^2 sigma model. Note that the Lax connection is defined as

$$J^A = a j^A + b * j^A, \tag{2.15}$$

where the Hodge dual is defined as

$$(*\mathrm{d}f)_\alpha = -\sqrt{-\gamma} \partial_\gamma f \gamma^{\gamma\delta} \epsilon_{\delta\alpha} \tag{2.16}$$

for any function f. Further, a, b given in (2.14) depend on spectral parameter Λ as

$$a = \tfrac{1}{2}[1 \pm \cosh \Lambda], \qquad b = \tfrac{1}{2} \sinh \Lambda \tag{2.17}$$

so that $a^2 - a - b^2 = 0$. Explicitly, for spatial components of J_σ^A we obtain

$$J_\sigma^A = a j_\sigma^A - b\sqrt{-\gamma}\gamma^{\tau\beta} j_\beta^A = a j_\sigma^A + b j_P^A. \tag{2.18}$$

Then using (2.11) and (2.14), we determine the Poisson brackets between the spatial components of Lax connections for two different spectral parameters Λ, Λ':

$$\left\{ J_\sigma^A(\sigma, \Lambda), J_\sigma^B(\sigma', \Lambda') \right\} = -ab' K^{AB} \partial_\sigma \delta(\sigma - \sigma') + ba' K^{AB} \partial_{\sigma'} \delta(\sigma - \sigma')$$
$$- ab' j_\sigma^D f_{DC}{}^A K^{CB} \delta(\sigma - \sigma') + ba' j_\sigma^D f_{DC}{}^B K^{CA} \delta(\sigma - \sigma')$$
$$- bb' j_P^C f_{CD}{}^A K^{DB} \delta(\sigma - \sigma'). \tag{2.19}$$

Comparing with equation (A.16), we find

$$\mathbf{B}^{AB} = ba' K^{AB}, \qquad \mathbf{C}^{AB} = -ab' K^{AB},$$
$$\mathbf{A}^{AB} = -ab' j_\sigma^D f_{DC}{}^A K^{CB} - ba' j_\sigma^D f_{DC}{}^A K^{CB} - bb' j_P^C f_{CD}{}^A K^{DB}. \tag{2.20}$$

Fortunately, in this particular case, we can rather easily guess the form of matrices r^{AB}, s^{AB}. In fact, let us presume that the right side of equation (2.19) can be written in the form

$$(r-s)^{DB} f_{DC}{}^A J_\sigma^C(\Lambda) + (r+s)^{AD} f_{DC}{}^B J_\sigma^C(\Lambda') + \partial_\sigma (r-s)^{AB} \delta(\sigma - \sigma') - 2s^{AB} \partial_\sigma \delta(\sigma - \sigma'). \tag{2.21}$$

Then comparing expressions proportional to $\partial_\sigma \delta(\sigma - \sigma')$ in (2.19) and (2.21), we obtain

$$s^{AB} = \tfrac{1}{2} K^{AB}(ab' + ba'). \tag{2.22}$$

Further, we presume that

$$(r-s)^{AB} = A K^{AB}, \qquad (r+s)^{AB} = B K^{AB}, \tag{2.23}$$

where A, B are constants. Inserting these expressions into (2.21) and comparing with the right side of (2.19), we find

$$B = -\frac{b^2 a'}{a'b - b'a}, \qquad A = -\frac{b'^2 a}{a'b - b'a} \tag{2.24}$$

and then using (2.22), we finally obtain

$$r^{AB} = -\frac{1}{2(a'b - b'a)}(b'^2 a^2 + a'^2 b^2 - 2b'^2 b^2). \tag{2.25}$$

It is important that in the case of a principal chiral model, the objects r^{AB}, s^{AB} are constants and depend on spectral parameters Λ, Λ' only. On the other hand, the situation is much more involved in the case of a T-dual sigma model.

3. Poisson brackets of Lax connection in T-dual theory on S^2

In this section, we determine the Poisson bracket of Lax connections in the sigma model that is related to the sigma model on S^2 by T-duality along the compact $U(1)$ isometry cycle. Since the procedure for deriving T-dual action from the sigma model action on S^2 is nicely described in the paper [46], we use the results derived there and immediately write the T-dual action

$$S = -\frac{1}{2} \int \left(d\Theta * d\Theta + \frac{1}{\sin^2 \Theta} d\tilde{\Phi} * d\tilde{\Phi} \right)$$
$$= -\frac{1}{2} \int d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta} \left(\partial_\alpha \Theta \partial_\beta \Theta + \frac{1}{\sin^2 \Theta} \partial_\alpha \tilde{\Phi} \partial_\beta \tilde{\Phi} \right), \tag{3.1}$$

where the dual variable $\tilde{\Phi}$ is related to Φ through the relation

$$d\tilde{\Phi} = \sin^2 \Theta * d\Phi. \tag{3.2}$$

While the original action was $SO(3)$ invariant, the manifest symmetry of the T-dual action (3.1) is simply the $U(1)$ shift of $\tilde{\Phi}$. As was argued in [46], the full $SO(3)$ symmetry group is hidden and it is realized non-locally.

As usual in the Hamiltonian formalism, we first determine from action (3.1) the momenta P_Θ, $P_{\tilde{\Phi}}$ conjugate to Θ, $\tilde{\Phi}$:

$$P_\Theta = -\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\Theta, \qquad P_{\tilde{\Phi}} = -\frac{1}{\sin^2\Theta}\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\tilde{\Phi} \tag{3.3}$$

with corresponding Poisson brackets

$$\{\tilde{\Phi}(\sigma), P_{\tilde{\Phi}}(\sigma')\} = \delta(\sigma - \sigma'), \qquad \{\Theta(\sigma), P_\Theta(\sigma')\} = \delta(\sigma - \sigma'). \tag{3.4}$$

To proceed further, we have to say few words considering the problem of how the Lax connection behaves under T-duality transformation. It is well known that T-duality transformation (3.2) cannot be directly performed on currents (2.15) and (2.18) since they depend not only on $d\Phi$, but also explicitly on coordinate Φ. A solution of this problem is based on the observation that for any $g \in G$, the new current

$$J' = g^{-1}Jg + g^{-1}\,dg \tag{3.5}$$

is again flat[6]. Then there exists an element $g \in SO(3)$ that transforms the original currents into new ones that depend on Φ only through its derivatives. The forms of the matrix g and corresponding current J' were found in [46] with the result

$$\begin{aligned}
J'^1 &= -\tfrac{1}{\sqrt{2}}\sin\Theta\cos\Theta(a\,d\Phi + b * d\Phi), \\
J'^2 &= -\tfrac{1}{\sqrt{2}}(a\,d\Theta + b * d\Theta), \\
J'^3 &= -\tfrac{1}{\sqrt{2}}\sin^2\Theta(a\,d\Phi + b * d\Phi) + \sqrt{2}\,d\Phi.
\end{aligned} \tag{3.6}$$

Then using (3.2) we define T-dual flat currents as $\hat{J}^A = J'^A(\Phi \to \tilde{\Phi})$. Explicitly, their spatial components take the form

$$\begin{aligned}
\tilde{J}^1_\sigma &= -\frac{1}{\sqrt{2}}\frac{\cos\Theta}{\sin\Theta}(-a\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\tilde{\Phi} + b\partial_\sigma\tilde{\Phi}), \\
\tilde{J}^2_\sigma &= -\frac{1}{\sqrt{2}}(a\partial_\sigma\Theta - b\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\Theta), \\
\tilde{J}^3_\sigma &= -\frac{1}{\sqrt{2}}(b\partial_\sigma\tilde{\Phi} - a\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\tilde{\Phi}) - \frac{\sqrt{2}}{\sin^2\Theta}\sqrt{-\gamma}\gamma^{\tau\beta}\partial_\beta\tilde{\Phi}
\end{aligned} \tag{3.7}$$

or alternatively as functions of phase space variables $(\Theta, P_\Theta, \tilde{\Phi}, P_{\tilde{\Phi}})$:

$$\begin{aligned}
\tilde{J}^1_\sigma &= -\frac{1}{\sqrt{2}}a\cos\Theta\sin\Theta P_{\tilde{\Phi}} + \frac{1}{\sqrt{2}}b\frac{\cos\Theta}{\sin\Theta}\partial_\sigma\tilde{\Phi}, \\
\tilde{J}^2_\sigma &= -\frac{1}{\sqrt{2}}a\partial_\sigma\Theta + \frac{1}{\sqrt{2}}bP_\Theta, \\
\tilde{J}^3_\sigma &= -\frac{1}{\sqrt{2}}b\partial_\sigma\tilde{\Phi} + \frac{a}{\sqrt{2}}\sin^2\Theta P_{\tilde{\Phi}} + \sqrt{2}P_{\tilde{\Phi}}.
\end{aligned} \tag{3.8}$$

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[6] We will discuss the general procedure in section 5.

Then using (3.4), we calculate the Poisson brackets of the spatial components of Lax connections for two spectral parameters Λ, Λ'. After some straightforward calculations, we obtain

$$\left\{\hat{J}_\sigma^A(\sigma, \Lambda), \hat{J}_\sigma^B(\sigma', \Lambda')\right\} = \mathbf{A}^{AB}(\sigma, \Lambda, \Lambda')\delta(\sigma - \sigma')\mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Lambda')\partial_{\sigma'}\delta(\sigma - \sigma')$$
$$+ \mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Lambda')\partial_\sigma\delta(\sigma - \sigma'), \tag{3.9}$$

where

$$\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Lambda') = \mathbf{A}^{AB}(\sigma, \Lambda, \Lambda')(T_A)_{\alpha\beta}(T_B)_{\gamma\delta}$$
$$= -\frac{1}{2}\left[P_{\tilde{\Phi}}ab'(\sin^2\Theta - \cos^2\Theta) + bb'\frac{1}{\sin^2\Theta}\partial_\sigma\tilde{\Phi}\right](T_1)_{\alpha\beta}(T_2)_{\gamma\delta}$$
$$+ \frac{1}{2}\left[P_{\tilde{\Phi}}ba'(\sin^2\Theta - \cos^2\Theta) + bb'\frac{1}{\sin^2\Theta}\partial_\sigma\tilde{\Phi}\right](T_2)_{\alpha\beta}(T_1)_{\gamma\delta}$$
$$- ba'\sin\Theta\cos\Theta(T_2)_{\alpha\beta}(T_3)_{\gamma\delta} + ab'\sin\Theta\cos\Theta(T_3)_{\alpha\beta}(T_2)_{\gamma\delta}, \tag{3.10}$$

$$\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Lambda') = \mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Lambda')(T_A)_{\alpha\beta}(T_B)_{\gamma\delta}$$
$$= \frac{1}{2}ba'\frac{\cos\Theta(\sigma)}{\sin\Theta(\sigma)}\cos\Theta(\sigma')\sin\Theta(\sigma')(T_1)_{\alpha\beta}(T_1)_{\gamma\delta}$$
$$+ \left(\frac{1}{2}ba'\frac{\cos\Theta(\sigma)}{\sin\Theta(\sigma)}\sin^2\Theta(\sigma') - b\frac{\cos\Theta(\sigma)}{\sin\Theta(\sigma)}\right)(T_1)_{\alpha\beta}(T_3)_{\gamma\delta}$$
$$+ \frac{1}{2}ab'(T_2)_{\alpha\beta}(T_2)_{\gamma\delta} + \frac{1}{2}ba'\cos\Theta(\sigma')\sin\Theta(\sigma')(T_3)_{\alpha\beta}(T_1)_{\gamma\delta}$$
$$+ \left(\frac{1}{2}ba'\sin^2\Theta(\sigma') - b\right)(T_3)_{\alpha\beta}(T_3)_{\gamma\delta} \tag{3.11}$$

and

$$\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Lambda') = \mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Lambda')(T_A)_{\alpha\beta}(T_B)_{\gamma\delta}$$
$$= -\frac{1}{2}ab'\cos\Theta(\sigma)\sin\Theta(\sigma)\frac{\cos\Theta(\sigma')}{\sin\Theta(\sigma')}(T_1)_{\alpha\beta}(T_1)_{\gamma\delta}$$
$$- \frac{1}{2}ab'\cos\Theta(\sigma)\sin\Theta(\sigma)(T_1)_{\alpha\beta}(T_3)_{\gamma\delta} - \frac{1}{2}a'b(T_2)_{\alpha\beta}(T_2)_{\gamma\delta}$$
$$+ \left(-\frac{1}{2}ab'\sin^2\Theta(\sigma)\frac{\cos\Theta(\sigma')}{\sin\Theta(\sigma')} + b'\frac{\cos\Theta(\sigma')}{\sin\Theta(\sigma')}\right)(T_3)_{\alpha\beta}(T_1)_{\gamma\delta}$$
$$+ \left(-\frac{1}{2}ab'\sin^2\Theta(\sigma) + b'\right)(T_3)_{\alpha\beta}(T_3)_{\gamma\delta}. \tag{3.12}$$

As a check, note that $\mathbf{A}^{AB}(\sigma, \Lambda, \Lambda')$, $\mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Lambda')$ and $\mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Lambda')$ obey the consistency relations (A.10). Using these results, we can partially determine the matrices r, s:

$$s_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Lambda')\frac{1}{2}(\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma, \Lambda, \Lambda') - \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma, \Lambda, \Lambda'))$$
$$= -\frac{1}{4}\left[(ab' + ba')\cos^2\Theta(T_1)_{\alpha\beta}(T_1)_{\gamma\delta} + (a'b + ba')(T_2)_{\alpha\beta}(T_2)_{\gamma\delta}\right.$$
$$+ \left((ab' + ba')\cos\Theta\sin\Theta - 2b\frac{\cos\Theta}{\sin\Theta}\right)(T_1)_{\alpha\beta}(T_3)_{\gamma\delta}$$

$$+\left((ab'+a'b)\cos\Theta\sin\Theta-2b'\frac{\cos\Theta}{\sin\Theta}\right)(T_3)_{\alpha\beta}(T_1)_{\gamma\delta}$$

$$+((ab'+ba')\sin^2\Theta-2(b'+b'))(T_3)_{\alpha\beta}(T_3)_{\gamma\delta}\Big] \tag{3.13}$$

and

$$
\begin{aligned}
r_{\alpha\gamma,\beta\delta}(\sigma,\Lambda,\Lambda') &= \tfrac{1}{2}[\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,\Lambda,\Lambda')+\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,\Lambda',\Lambda)]+\hat{r}_{\alpha\gamma,\beta\delta}(\sigma,w,v)\\
&= -\tfrac{1}{4}[(ab'-ba')\cos^2\Theta(T_1)_{\alpha\beta}(T_1)_{\gamma\delta}+(ab'-ba')\cos\Theta\sin\Theta(T_1)_{\alpha\beta}(T_3)_{\gamma\delta}\\
&\quad -(ab'-a'b)\cos\Theta\sin\Theta(T_3)_{\alpha\beta}(T_1)_{\gamma\delta}+(a'b-ba')(T_2)_{\alpha\beta}(T_2)_{\gamma\delta}\\
&\quad +((ab'-ba')\sin^2\Theta-2(b'-b))(T_3)_{\alpha\beta}(T_3)_{\gamma\delta}]\\
&\quad +\hat{r}^{AB}(\sigma,\Lambda,\Lambda')(T_A)_{\alpha\beta}(T_B)_{\gamma\delta},
\end{aligned}
\tag{3.14}
$$

where $\hat{r}^{AB}$ is a solution of the differential equation (A.13). Unfortunately, due to the fact that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ explicitly depend on the phase space variables it is very difficult to solve this differential equation (A.13) and we were not able to find an explicit form of $\hat{r}^{AB}$. On the other hand, it is important to stress that the Poisson brackets of Lax connections take the form as in (A.9) and hence following the arguments given in the appendix, we can argue that the T-dual sigma model contains an infinite number of conserved charges that are in involution in the sense that their Poisson brackets vanish. In summary, T-dual theory is classically integrable as well in spite of the fact that the Poisson bracket structure is intricate.

4. Second example: T-dual AdS$_2$ string

As the second example of T-dual theory, we consider the case of a bosonic sigma model on AdS$_2$ and its T-dual version. Recall that the dynamics of the bosonic string on AdS$_2$ is governed by an action

$$S=-\frac{1}{2}\int d^2\sigma\frac{1}{Y^2}\sqrt{-\gamma}\gamma^{\alpha\beta}(\partial_\alpha X\partial_\beta X+\partial_\alpha Y\partial_\beta Y). \tag{4.1}$$

With analogy with the previous section, we introduce three currents:

$$
\begin{aligned}
j_\alpha^1 &= \frac{1}{2\sqrt{2}Y^2}((1+(X^2-Y^2))\partial_\alpha X+2XY\partial_\alpha Y),\\
j_\alpha^2 &= \frac{1}{2\sqrt{2}Y^2}((1-(X^2-Y^2))\partial_\alpha X-2XY\partial_\alpha Y),\\
j_\alpha^3 &= -\frac{1}{\sqrt{2}Y^2}(X\partial_\alpha X+Y\partial_\alpha Y)
\end{aligned}
\tag{4.2}
$$

that are conserved

$$\partial_\alpha[\sqrt{-\gamma}\gamma^{\alpha\beta}j_{\beta A}]=0, \qquad A=1,2,3. \tag{4.3}$$

Further, it can be shown that these currents are flat:

$$\partial_\alpha j_\beta^A-\partial_\beta j_\alpha^A+j_\alpha^B j_\beta^C f_{BC}{}^A=0, \tag{4.4}$$

where $f_{BC}{}^A=-\frac{1}{\sqrt{2}}\epsilon_{BCD}K^{DA}$ and the Cartan–Killing form K^{AB} is equal to $K^{AB}=\text{diag}(-1,1,1)$. Then it is easy to see that the sigma model action (4.1) can be expressed as a principal chiral model with the corresponding Lax connection

$$J=aj+b*j, \qquad a=\tfrac{1}{2}[1\pm\cosh\Lambda], \qquad b=\tfrac{1}{2}\sinh\Lambda. \tag{4.5}$$

Our goal is to develop the Hamiltonian formalism for T-dual theory where T-duality is performed along the X direction [46] so that the T-dual action takes the form

$$S = -\frac{1}{2}\int d^2\sigma \sqrt{-\gamma}\left[\frac{1}{\tilde{Y}^2}\gamma^{\alpha\beta}\partial_\alpha\tilde{X}\partial_\beta\tilde{X} + \frac{1}{\tilde{Y}^2}\gamma^{\alpha\beta}\partial_\alpha\tilde{Y}\partial_\beta\tilde{Y}\right], \tag{4.6}$$

where we also introduced $\tilde{Y}$ defined as

$$\tilde{Y} = \frac{1}{Y}. \tag{4.7}$$

It is clear that action (4.6) again describes the dynamics of string on the AdS$_2$ background and hence the Lax connection for given theory is the same as the original one (4.5) when we replace X and Y with $\tilde{X}$ and $\tilde{Y}$. On the other hand, there exists a Lax connection in the T-dual background that is related to the original Lax connection by gauge transformations and then by substitutions $X, Y \to \tilde{X}, \tilde{Y}$. This Lax connection was derived in [46] and takes the form

$$\hat{J}^1_\alpha = -\frac{1}{2\sqrt{2}}\frac{(1-\tilde{Y}^2)}{\tilde{Y}^2}(-a\partial_\gamma\tilde{X}\sqrt{-\gamma}\gamma^{\gamma\delta}\epsilon_{\delta\alpha} + b\partial_\alpha\tilde{X}) - \sqrt{2}\frac{1}{\tilde{Y}^2}\partial_\gamma\tilde{X}\sqrt{-\gamma}\gamma^{\gamma\delta}\epsilon_{\delta\alpha},$$

$$\hat{J}^2_\alpha = \frac{1}{\tilde{Y}\sqrt{2}}(a\partial_\alpha\tilde{Y} - b\partial_\gamma\tilde{Y}\sqrt{-\gamma}\gamma^{\gamma\delta}\epsilon_{\delta\alpha}), \tag{4.8}$$

$$\hat{J}^3_\alpha = -\frac{1}{2\sqrt{2}}\frac{(1+\tilde{Y}^2)}{\tilde{Y}^2}(a\partial_\gamma\tilde{X}\sqrt{-\gamma}\gamma^{\gamma\delta}\epsilon_{\delta\alpha} - b\partial_\alpha\tilde{X}) + \sqrt{2}\frac{1}{\tilde{Y}^2}\partial_\gamma\tilde{X}\sqrt{-\gamma}\gamma^{\gamma\delta}\epsilon_{\delta\alpha}.$$

To proceed further, we derive from (4.6) the conjugate momenta

$$P_{\tilde{X}} = -\frac{1}{\tilde{Y}^2}\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\tilde{X}, \qquad P_{\tilde{Y}} = -\frac{1}{\tilde{Y}^2}\sqrt{-\gamma}\gamma^{\tau\alpha}\partial_\alpha\tilde{Y}. \tag{4.9}$$

Then the spatial components of Lax connection expressed as functions of canonical variables are equal to

$$\hat{J}^1_\sigma = -\frac{1}{2\sqrt{2}}\frac{(1-\tilde{Y}^2)}{\tilde{Y}^2}(a\tilde{Y}^2 P_{\tilde{X}} + b\partial_\sigma\tilde{X}) + \sqrt{2}P_{\tilde{X}},$$

$$\hat{J}^2_\sigma = \frac{1}{\sqrt{2}\tilde{Y}}(a\partial_\sigma\tilde{Y} + b\tilde{Y}^2 P_{\tilde{Y}}), \tag{4.10}$$

$$\hat{J}^3_\sigma = -\frac{1}{2\sqrt{2}}\frac{(1+\tilde{Y}^2)}{\tilde{Y}^2}(a\tilde{Y}^2 P_{\tilde{X}} + b\partial_\sigma\tilde{X}) - \sqrt{2}P_{\tilde{X}}.$$

Now we are ready to determine the Poisson bracket of spatial components of the Lax connection. Again, after some calculations we derive the Poisson brackets that have the same form as in (A.5) where the matrices **A**, **C** and **B** are equal to

$$\begin{aligned}
\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Lambda') &= \left[\frac{b'}{2\tilde{Y}^2}(a\tilde{Y}^2 P_{\tilde{X}} + b\partial_\sigma\tilde{X}) - \frac{1}{2}ab'(1-\tilde{Y}^2)\tilde{Y}^2 P_{\tilde{X}}\right](T^1)_{\alpha\beta}(T^2)_{\gamma\delta} \\
&\quad + \left[-\frac{b}{2\tilde{Y}^2}(a'\tilde{Y}^2 P_{\tilde{X}} + b'\partial_\sigma\tilde{X}) + \frac{1}{2}ba'(1-\tilde{Y}^2)\tilde{Y}^2 P_{\tilde{X}}\right](T^2)_{\alpha\beta}(T^1)_{\gamma\delta} \\
&\quad + \frac{1}{2\tilde{Y}}(ba' - ab')\partial_\sigma\tilde{Y}(T^2)_{\alpha\beta}(T^2)_{\gamma\delta} \\
&\quad + \left[\frac{b}{2\tilde{Y}^2}(a'\tilde{Y}^2 P_{\tilde{X}} + b'\partial_\sigma\tilde{X}) + \frac{1}{2}ba'(1+\tilde{Y}^2)P_{\tilde{X}}\right](T^2)_{\alpha\beta}(T^3)_{\gamma\delta} \\
&\quad - \left[\frac{b'}{2\tilde{Y}^2}(a\tilde{Y}^2 P_{\tilde{X}} + b\partial_\sigma\tilde{X}) + \frac{1}{2}ab'(1+\tilde{Y}^2)P_{\tilde{X}}\right](T^3)_{\alpha\beta}(T^2)_{\gamma\delta}
\end{aligned} \tag{4.11}$$

9

and

$$
\begin{aligned}
\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',\Lambda,\Lambda') ={}& \left[\frac{ba'}{8}\frac{(1-\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}(1-Y^2(\sigma'))-\frac{b}{2}\frac{(1-\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}\right](T^1)_{\alpha\beta}(T^1)_{\gamma\delta} \\
&+\left[\frac{ba'}{8}\frac{(1-\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}(1+\tilde{Y}^2(\sigma'))+\frac{b}{2}\frac{(1-\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}\right](T^1)_{\alpha\beta}(T^3)_{\gamma\delta} \\
&+\frac{ab'}{2}\frac{\tilde{Y}^2(\sigma')}{\tilde{Y}(\sigma)}(T^2)_{\alpha\beta}(T^2)_{\gamma\delta} \\
&+\left[\frac{ba'}{8}\frac{(1+\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}(1-\tilde{Y}^2(\sigma'))-\frac{b}{2}\frac{1+\tilde{Y}^2(\sigma)}{\tilde{Y}^2(\sigma)}\right](T^3)_{\alpha\beta}(T^1)_{\gamma\delta} \\
&+\left[\frac{ba'}{8}\frac{(1+\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}(1+\tilde{Y}^2(\sigma'))-\frac{b}{2}\frac{(1+\tilde{Y}^2(\sigma))}{\tilde{Y}^2(\sigma)}\right](T^3)_{\alpha\beta}(T^3)_{\gamma\delta}, \\
\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',\Lambda,\Lambda') ={}& -\left[\frac{ab'}{8}(1-\tilde{Y}^2(\sigma)\frac{(1-\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}-\frac{1}{2}\frac{(1-\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}b'\right](T^1)_{\alpha\beta}(T^1)_{\gamma\delta} \\
&-\left[\frac{ab'}{8}(1-\tilde{Y}^2(\sigma))\frac{(1+\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}-\frac{b'}{2}\frac{(1+\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}\right](T^1)_{\alpha\beta}(T^3)_{\gamma\delta} \\
&-\frac{ba'}{2}\frac{\tilde{Y}^2(\sigma)}{\tilde{Y}(\sigma')}(T^2)_{\alpha\beta}(T^2)_{\gamma\delta} \\
&-\left[\frac{ab'}{8}(1+\tilde{Y}^2(\sigma))\frac{(1-\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}+\frac{b'}{2}\frac{(1-\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}\right](T^3)_{\alpha\beta}(T^1)_{\gamma\delta} \\
&-\left[\frac{a'b}{8}(1+\tilde{Y}^2(\sigma))\frac{(1+\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}-\frac{b'}{2}\frac{(1+\tilde{Y}^2(\sigma'))}{\tilde{Y}^2(\sigma')}\right](T^3)_{\alpha\beta}(T^3)_{\gamma\delta}.
\end{aligned}
\tag{4.12}
$$

As a check, note that matrices (4.11) and (4.12) obey the consistency relations (A.6). Further, we can also determine the matrix s^{AB}; however, we are not able to fully determine r^{AB} due to the fact that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are functions of phase space variables. It is clear that the theory is classically integrable since we can in principle find an infinite number of charges that are in involutions. However, the consequence of the non-local nature of the T-dual Lax connection is that the matrices r, s now explicitly depend on phase space variables and are non-local. On the other hand, the case of AdS$_2$ is exceptional since we know that its T-dual image is again AdS$_2$ so that we can find Lax connection corresponding to the standard principal chiral with constant r and s matrices. We are not going to study the relations between these two Lax connections in this paper. We hope to return to the study of this problem in a future publication.

5. General procedure

In this section, we consider a general situation when we have a principal chiral model with a field $g(\sigma)$ that maps the string worldsheet into some group G with Lie algebra $\mathbf{g}$. Further, we presume that the Lie algebra $\mathbf{g}$ has generators T_A, $A = 1, \ldots, \dim(\mathbf{g})$, that obey the relation

$$
[T_A, T_B] = f_{AB}{}^C T_C. \tag{5.1}
$$

From $g(x)$, we can construct a current j in the form

$$
j = g^{-1} \, \mathrm{d}g \equiv E_M^A \, \mathrm{d}x^M T_A, \tag{5.2}
$$

where by definition

$$\mathrm{d}j + j \wedge j = 0, \tag{5.3}$$

and where we introduced sigma model coordinates x^M. Then the dynamics of the theory is governed by the action

$$S = -\frac{1}{2} \int \mathrm{d}^2\sigma \sqrt{-\gamma}\, \gamma^{\alpha\beta} K_{AB} E_M^A \partial_\alpha x^M E_N^B \partial_\beta x^N, \tag{5.4}$$

where $K_{AB} = \mathrm{Tr}(T_A T_B)$. As we reviewed in section 2, the principal chiral model possesses Lax connection $J = aj + b * j$ that is flat:

$$\mathrm{d}J + J \wedge J = 0 \tag{5.5}$$

for $a = \frac{1}{2}[1 \pm \cosh \Lambda]$ and $b = \frac{1}{2} \sinh \Lambda$, where Λ is a spectral parameter.

The principal chiral model has an important property that when we perform the gauge transformation from $g \in G$ on the original *Lax connection*

$$J' = g^{-1} J g + g^{-1} \mathrm{d}g, \tag{5.6}$$

we obtain the fact that the new one is again flat:

$$\mathrm{d}J' + J' \wedge J' = g^{-1}(\mathrm{d}J + J \wedge J)g = 0. \tag{5.7}$$

To proceed further, we write the gauge transformation (5.6) in the component formalism. Since $J' = J'^A T_A$, we obtain

$$J'^A = J^C \Omega_C^A + e^A, \tag{5.8}$$

where

$$g^{-1}\,\mathrm{d}g = e^A T_A, \qquad \Omega_{CB} = \mathrm{Tr}(g^{-1} T_C g T_B), \qquad \Omega_C^A = \Omega_{CB} K^{BA}, \tag{5.9}$$

where generally Ω_C^A and e^A are functions of phase space variables. Our goal is to determine the Poisson bracket of the Lax connection in T-dual theory. The first step in this direction is to determine the Poisson bracket of the Lax connection J'. Using (5.8), we obtain

$$
\begin{aligned}
\left\{ J'^A_\sigma(\sigma, \Lambda)\,,\, J'^B_\sigma(\sigma', \Gamma) \right\} = {}& \left\{ e^A_\sigma(\sigma), e^B_\sigma(\sigma') \right\} \left\{ e^A_\sigma(\sigma), J^C_\sigma(\Gamma, \sigma') \right\} \Omega_C^B(\sigma') \\
& + \left\{ e^A_\sigma(\sigma), \Omega_C^B(\sigma') \right\} J^C_\sigma(\Gamma, \sigma') + \left\{ J^C_\sigma(\Lambda, \sigma), e^B_\sigma(\sigma') \right\} \Omega_C^A(\sigma) \\
& + J^C_\sigma(\Lambda, \sigma) \left\{ \Omega_C^A(\sigma), e^B_\sigma(\sigma') \right\} + \Omega_C^A(\sigma) \left\{ J^C_\sigma(\Lambda, \sigma), J^D_\sigma(\Gamma, \sigma') \right\} \Omega_D^B(\sigma') \\
& + \left\{ J^C_\sigma(\Lambda, \sigma), \Omega_D^B(\sigma') \right\} \Omega_C^A(\sigma) J^D_\sigma(\Gamma, \sigma') \\
& + J^C_\sigma(\Lambda, \sigma) \left\{ \Omega_C^A(\sigma), J^D_\sigma(\Gamma, \sigma') \right\} \Omega_D^B(\sigma') \\
& + J^C_\sigma(\sigma, \Lambda) \left\{ \Omega_C^A(\sigma), \Omega_D^B(\sigma') \right\} J^B_\sigma(\sigma', \Gamma).
\end{aligned} \tag{5.10}
$$

Let us now presume that g is a function of x^M only. Then the spatial component e^A_σ depends on x^M and their derivatives x^M and does not depend on p_M. It is also clear that Ω_A^B depends on x^M only. Then we obtain

$$\left\{ e^A_\sigma(\sigma), e^B_\sigma(\sigma') \right\} = 0, \qquad \left\{ e^A_\sigma(\sigma), \Omega_B^C(\sigma') \right\} = 0, \qquad \left\{ \Omega_A^B(\sigma), \Omega_C^D(\sigma') \right\} = 0. \tag{5.11}$$

Then, using the above arguments and the fact that J^C_σ is linear in momenta we can presume that

$$
\begin{aligned}
\left\{ e^A_\sigma(\sigma), J^B_\sigma(\sigma', \Lambda) \right\} = {}& \mathcal{E}^{AB}(\sigma, \sigma', \Lambda) \partial_\sigma \delta(\sigma - \sigma') \\
& + \mathcal{F}^{AB}(\sigma, \sigma', \Lambda) \partial_{\sigma'} \delta(\sigma - \sigma') + \mathcal{G}^{AB}(\sigma, \Lambda) \delta(\sigma - \sigma'),
\end{aligned} \tag{5.12}
$$

$$\left\{ \Omega_A^B(\sigma), J^C_\sigma(\Gamma, \sigma') \right\} = \tilde{\Omega}_A^{BC}(\sigma, \Gamma) \delta(\sigma - \sigma').$$

Further, let us presume that Poisson brackets between $J_\sigma^A(\sigma, \Lambda)$ and $J_\sigma^B(\sigma', \Gamma)$ take the form

$$\left\{ J_\sigma^A(\sigma, \Lambda), J_\sigma^B(\sigma', \Gamma) \right\} = \mathbf{A}^{AB}(\sigma, \Lambda, \Gamma)\delta(\sigma - \sigma')$$
$$+ \mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_{\sigma'}\delta(\sigma - \sigma') + \mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma'). \qquad (5.13)$$

Then if we insert this expression into (5.10), we obtain the fact that the Poisson bracket of Lax connections J' has the same form as (5.13):

$$\left\{ J_\sigma'^A(\sigma, \Lambda), J_\sigma'^B(\sigma', \Gamma) \right\} = \mathbf{A}'^{AB}(\sigma, \Lambda, \Gamma)\delta(\sigma - \sigma')$$
$$+ \mathbf{C}'^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma') + \mathbf{B}'^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_{\sigma'}\delta(\sigma - \sigma'), \qquad (5.14)$$

where

$$\begin{aligned}
\mathbf{A}'^{AB}(\sigma, \Lambda, \Gamma) &= \mathcal{G}^{AC}(\sigma, \Gamma)\Omega_C^B(\sigma) - \Omega_C^A(\sigma)\mathcal{G}^{BC}(\sigma, \Lambda) + \Omega_C^A(\sigma)\mathbf{A}^{CD}(\sigma, \Lambda, \Gamma)\Omega_D^B(\sigma) \\
&\quad - J_\sigma^D(\Gamma, \sigma)\tilde{\Omega}_D^{BC}(\sigma, \Lambda)\Omega_C^A(\sigma) + J_\sigma^C(\Lambda, \sigma)\tilde{\Omega}_C^{DA}(\sigma, \Gamma)\Omega_D^B(\sigma), \\
\mathbf{B}'^{AB}(\sigma, \sigma', \Lambda, \Gamma) &= \mathcal{F}^{AC}(\sigma, \sigma', \Gamma)\Omega_C^B(\sigma') - \Omega_C^A(\sigma)\mathcal{E}^{BC}(\sigma', \sigma, \Lambda) \\
&\quad + \Omega_C^A(\sigma)\mathbf{B}^{CD}(\sigma, \sigma', \Lambda, \Gamma)\Omega_D^B(\sigma'), \\
\mathbf{C}'^{AB}(\sigma, \sigma', \Lambda, \Gamma) &= \mathcal{E}^{AC}(\sigma, \sigma', \Gamma)\Omega_C^B(\sigma') - \Omega_C^A(\sigma)\mathcal{F}^{BC}(\sigma', \sigma, \Lambda) \\
&\quad + \Omega_C^A(\sigma)\mathbf{C}^{CD}(\sigma, \sigma', \Lambda, \Gamma)\Omega_D^B(\sigma').
\end{aligned} \qquad (5.15)$$

Now we are ready to calculate the Poisson brackets of Lax connection in T-dual theory. Let us denote $\eta^I(\sigma) = (x^1(\sigma), \dots, x^n(\sigma), p_1(\sigma), \dots, p_n(\sigma))$, $I = 1, \dots, 2n$, where n is the dimension of M, and introduce a symplectic structure $\mathbf{J}^{IJ}(\sigma, \sigma')$ defined as

$$\mathbf{J} = \begin{pmatrix} 0 & I_{n\times n}\delta(\sigma - \sigma') \\ -I_{n\times n}\delta(\sigma - \sigma') & 0 \end{pmatrix}. \qquad (5.16)$$

Then the Poisson bracket of two functions $F(\eta)$, $G(\eta)$ can be written as

$$\{F, G\}_\eta = \int \mathrm{d}\sigma \, \mathrm{d}\sigma' \left(\frac{\delta F}{\delta \eta^I(\sigma)} \mathbf{J}^{IJ}(\sigma, \sigma') \frac{\delta G}{\delta \eta^J(\sigma')} \right). \qquad (5.17)$$

For example, if $F = x^N(\sigma)$, $G = p_M(\sigma')$, we obtain

$$\{x^N(\sigma), p_M(\sigma')\} = \delta_M^N \delta(\sigma - \sigma') \qquad (5.18)$$

or more covariantly

$$\{\eta^I(\sigma), \eta^J(\sigma')\}_\eta = \mathbf{J}^{IJ}(\sigma, \sigma'). \qquad (5.19)$$

An important property of T-duality is that it is a sort of canonical transformation. More precisely, let us denote the variables in T-dual theory as $\tilde{\eta}^I = (\tilde{x}^M(\sigma), \tilde{p}_M(\sigma))$ and presume that they can be expressed as functions of original variables:

$$\tilde{\eta}^I(\sigma) = \tilde{\eta}^I(\eta(\sigma)). \qquad (5.20)$$

Now if the transformation from η to $\tilde{\eta}$ is canonical, we have the fact that the matrix

$$M_J^I(\sigma, \sigma') = \frac{\delta \tilde{\eta}^I(\sigma)}{\delta \eta^J(\sigma')} \qquad (5.21)$$

preserves the symplectic structure in the sense that

$$\int \mathrm{d}x \, \mathrm{d}y \, M_K^I(\sigma, x)\mathbf{J}^{KL}(x, y)M_L^J(y, \sigma') = \mathbf{J}^{IJ}(\sigma, \sigma'). \qquad (5.22)$$

Using this fact, we immediately obtain

$$\{\tilde{\eta}^I(\sigma), \tilde{\eta}^J(\sigma)\}_{\tilde{\eta}} = \mathbf{J}^{IJ}(\sigma, \sigma). \qquad (5.23)$$

This expression implies that all Poisson brackets are invariant under canonical transformations. In fact, let us consider two functionals $F(\eta)$ and $G(\eta)$. Then the invariance of the Poisson bracket under the canonical transformation (5.23) implies

$$\{F(\eta), G(\eta)\}_\eta = \{F(\tilde{\eta}), G(\tilde{\eta})\}_{\tilde{\eta}}, \tag{5.24}$$

where η and $\tilde{\eta}$ are related by canonical transformation. Then if we apply these considerations to the case of Lax connections in original and T-dual theories, we obtain

$$
\begin{aligned}
\left\{ \hat{J}_\sigma^A(\tilde{\eta}(\sigma), \Lambda), \hat{J}_\sigma^B(\tilde{\eta}(\sigma'), \Gamma) \right\}_{\tilde{\eta}} &= \left\{ \hat{J}_\sigma^A(\Lambda, \eta(\sigma)), \hat{J}_\sigma^B(\eta(\sigma'), \Gamma) \right\}_\eta \\
&= \mathbf{A}'^{AB}(\eta(\sigma), \Lambda, \Gamma)\delta(\sigma - \sigma') + \mathbf{C}'^{AB}(\eta(\sigma), \eta(\sigma'), \Lambda, \Gamma)\partial_\sigma \delta(\sigma - \sigma') \\
&\quad + \mathbf{B}'^{AB}(\eta(\sigma), \eta(\sigma'), \Lambda, \Gamma)\partial_{\sigma'} \delta(\sigma - \sigma') \\
&= \tilde{\mathbf{A}}^{AB}(\tilde{\eta}(\sigma), \Lambda, \Gamma)\delta(\sigma - \sigma') + \tilde{\mathbf{C}}^{AB}(\tilde{\eta}(\sigma), \tilde{\eta}(\sigma'), \Lambda, \Gamma)\partial_\sigma \delta(\sigma - \sigma') \\
&\quad + \tilde{\mathbf{B}}^{AB}(\tilde{\eta}(\sigma), \tilde{\eta}(\sigma'), \Lambda, \Gamma)\partial_{\sigma'} \delta(\sigma - \sigma'),
\end{aligned}
\tag{5.25}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{A}}^{AB}(\tilde{\eta}(\sigma), \Lambda, \Gamma) &\equiv \mathbf{A}'^{AB}(\eta(\tilde{\eta}(\sigma)), \Lambda, \Gamma), \\
\tilde{\mathbf{B}}^{AB}(\tilde{\eta}(\sigma), \tilde{\eta}(\sigma'), \Lambda, \Gamma) &\equiv \mathbf{B}'^{AB}(\eta(\tilde{\eta}(\sigma)), \eta(\tilde{\eta}(\sigma')), \Lambda, \Gamma), \\
\tilde{\mathbf{C}}^{AB}(\tilde{\eta}(\sigma), \tilde{\eta}(\sigma'), \Lambda, \Gamma) &\equiv \mathbf{C}'^{AB}(\eta(\tilde{\eta}(\sigma)), \eta(\tilde{\eta}(\sigma')), \Lambda, \Gamma),
\end{aligned}
\tag{5.26}
$$

and where in the first step we used (5.14) and in the second one we expressed (5.15) as functions of $\tilde{\eta}$. This result implies that we can express the Poisson bracket of Lax connections in T-dual theory using the known form of the original Lax connection J_σ^A and the known form of e_σ^A, Ω_A^B that are, in the final step, expressed as functions of T-dual variables. Then it is clear from (5.15) that the dual theory is again classically integrable in the sense that there is an infinite number of integrals of motion that are in involution. On the other hand, the complicated form of matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ implies that generally the matrices $\tilde{s}$ and $\tilde{r}$ are functions of phase space variables. Moreover, we can also expect that $\tilde{r}$ is non-local due to the fact that $\tilde{\tilde{r}}$ is a solution of the differential equation (A.13) with tilded functions $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$.

Acknowledgments

Appendix. Review of the basic properties of the monodromy matrix

In this appendix, we review the properties of the monodromy matrix, following [59, 60]. The monodromy matrix $\mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda)$, where Λ is a spectral parameter and α, β correspond to matrix indices, can be defined as

$$
\begin{aligned}
\partial_\sigma \mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda) &= \mathcal{A}_{\alpha\gamma}(\sigma, \Lambda)\mathcal{T}_{\gamma\beta}(\sigma, \sigma', \Lambda), \\
\partial_{\sigma'} \mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda) &= -\mathcal{T}_{\alpha\gamma}(\sigma, \sigma', \Lambda)\mathcal{A}_{\gamma\beta}(\sigma', \Lambda)
\end{aligned}
\tag{A.1}
$$

with the normalization condition

$$\mathcal{T}_{\alpha\beta}(\sigma, \sigma, \Lambda) = \delta_{\alpha\beta} \tag{A.2}$$

and

$$\mathcal{T}_{\alpha\beta}^{-1}(\sigma, \sigma', \Lambda) = \mathcal{T}_{\alpha\beta}(\sigma', \sigma, \Lambda). \tag{A.3}$$

Note that in our notation, $\mathcal{A}_{\alpha\beta}(\sigma, \Lambda)$ is the spatial component of the Lax connection.

The main interest in the theory of integrable systems is the Poisson bracket between $\mathcal{T}(\Lambda)$ and $\mathcal{T}(\Gamma)$. As was shown in a nice way in [59], the Poisson bracket between $\mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda)$ and $\mathcal{T}_{\gamma\delta}(\xi, \xi', \Gamma)$, where all $\sigma, \sigma', \xi, \xi'$ are distinct, is equal to

$$\{\mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda), \mathcal{T}_{\gamma\delta}(\xi, \xi', \Gamma)\} = \int_\sigma^{\sigma'} d\sigma_1 \int_\xi^{\xi'} d\xi_1 \mathcal{T}_{\alpha\sigma_1}(\sigma, \sigma_1, \Lambda)\mathcal{T}_{\gamma\rho_1}(\xi, \xi_1, \Gamma)$$
$$\times \left\{\mathcal{A}_{\sigma_1\sigma_2}(\sigma_1, \Lambda), \mathcal{A}_{\rho_1\rho_2}(\xi_1, \Gamma)\right\} \mathcal{T}_{\sigma_2\beta}(\sigma_1, \sigma', \Lambda)\mathcal{T}_{\rho_2\delta}(\xi_1, \xi', \Gamma). \quad (A.4)$$

The above result suggests that the fundamental role in the theory of integrable systems is the role of a Poisson bracket between the spatial components of the Lax connection. Let us now presume that the Poisson bracket of the spatial components of the Lax connection $\mathcal{A}(\sigma, \Lambda)$ and $\mathcal{A}(\sigma', \Gamma)$ takes the form

$$\{\mathcal{A}_{\alpha\beta}(\sigma, \Lambda), \mathcal{A}_{\gamma\delta}(\sigma', \Gamma)\} = \mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma)\delta(\sigma - \sigma')$$
$$+ \mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Gamma)\partial_{\sigma'}\delta(\sigma - \sigma') + \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma'), \quad (A.5)$$

where due to the antisymmetric property of Poisson brackets the functions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ obey consistency relations

$$\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) = -\mathbf{A}_{\gamma\alpha,\delta\beta}(\sigma, \Gamma, \Lambda),$$
$$\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Gamma) = -\mathbf{C}_{\gamma\alpha,\beta\delta}(\sigma', \sigma, \Gamma, \Lambda), \quad (A.6)$$
$$\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Gamma) = -\mathbf{B}_{\gamma\alpha,\delta\beta}(\sigma', \sigma, \Gamma, \Lambda).$$

Further, let us presume that the Lax connection takes a value in the Lie algebra **g** of some group G. Let us then presume that the generators of the algebra **g** are T_A, $A = 1, \ldots, \dim(G)$, with the following structure: $[T_A, T_B] = f_{AB}{}^C T_C$. Then we can write $\mathcal{A}$ in the form

$$\mathcal{A}(\sigma, \Lambda) = J_\sigma^A(\sigma, \Lambda)T_A \quad (A.7)$$

and also

$$\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) = \mathbf{A}^{AB}(\sigma, \Lambda, \Gamma)(T_A)_{\alpha\beta}(T_B)_{\gamma\delta},$$
$$\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma'\Lambda, \Gamma) = \mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Gamma)(T_A)_{\alpha\beta}(T_B)_{\gamma\delta}, \quad (A.8)$$
$$\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma', \Lambda, \Gamma) = \mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Gamma)(T_A)_{\alpha\beta}(T_B)_{\gamma\delta}.$$

Then the Poisson bracket (A.5) can be written as

$$\left\{J_\sigma^A(\sigma, \Lambda), J_\sigma^B(\sigma', \Gamma)\right\} = \mathbf{A}^{AB}(\sigma, \Lambda, \Gamma)\delta(\sigma - \sigma')$$
$$+ \mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_{\sigma'}\delta(\sigma - \sigma') + \mathbf{C}^{AB}(\sigma, \sigma', \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma') \quad (A.9)$$

and relations (A.6) take an alternative form:

$$\mathbf{A}^{AB}(\sigma, \Lambda, \Gamma) = -\mathbf{A}^{BA}(\sigma, \Gamma, \Lambda),$$
$$\mathbf{B}^{AB}(\sigma, \sigma', \Lambda, \Gamma) = -\mathbf{C}^{BA}(\sigma', \sigma, \Gamma, \Lambda). \quad (A.10)$$

Let us introduce the matrices $r_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma), s_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma)$ defined as

$$s_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) = \tfrac{1}{2}(\mathbf{B}_{\alpha\gamma,\beta,\delta}(\sigma, \sigma, \Lambda, \Gamma) - \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma, \Lambda, \Gamma))$$
$$r_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) = \tfrac{1}{2}(\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma, \sigma, \Lambda, \Gamma) + \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma, \sigma, \Lambda, \Gamma)) + \hat{r}_{\alpha\gamma,\beta\delta}(\sigma, w, v), \quad (A.11)$$

or alternatively

$$s^{AB}(\sigma, \Lambda, \Gamma) = \tfrac{1}{2}(\mathbf{B}^{AB}(\sigma, \sigma, \Lambda, \Gamma) - \mathbf{C}^{AB}(\sigma, \sigma, \Lambda, \Gamma))$$
$$= s^{BA}(\sigma, \Gamma, \Lambda)$$
$$r^{AB}(\sigma, \Lambda, \Gamma) = \tfrac{1}{2}(\mathbf{B}^{AB}(\sigma, \sigma, \Lambda, \Gamma) + \mathbf{C}^{AB}(\sigma, \sigma, \Lambda, \Gamma)) + \hat{r}^{AB}(\sigma, \Lambda, \Gamma)) \quad (A.12)$$
$$= -r^{BA}(\sigma, \Gamma, \Lambda),$$

where $\hat{r}^{AB}(\sigma, \Lambda, \Gamma)$ is a solution of the inhomogeneous first-order differential equation:

$$\partial_\sigma \hat{r}^{AB} + \hat{r}^{DB} f_{DC}{}^A J_\sigma^D(\Lambda) + \hat{r}^{AD} f_{DC}{}^B J_\sigma^C(\Gamma) = \Omega^{AB}, \tag{A.13}$$

where

$$\Omega^{AB}(\sigma, \Lambda, \Gamma) = \mathbf{A}^{AB}(\sigma, \Lambda, \Gamma) - \partial_u (\mathbf{B}(\sigma, u, \Lambda, \Gamma) + \mathbf{C}(u, \sigma, \Lambda, \Gamma))^{AB}_{u=\sigma}$$
$$- \mathbf{B}^{AC}(\sigma, \sigma, \Lambda, \Gamma) f_{CD}{}^B J_\sigma^D(\sigma, \Gamma) + J_\sigma^C(\sigma, \Lambda) f_{CD}{}^A \mathbf{C}^{DB}(\sigma, \sigma, \Lambda, \Gamma). \tag{A.14}$$

The significance of matrices s and r is that with their help, we can write the Poisson bracket (A.5) in the form [59]

$$\{\mathcal{A}_{\alpha\beta}(\sigma, \Lambda), \mathcal{A}_{\gamma\delta}(\sigma', \Gamma)\} = (\partial_\sigma r_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) - \partial_\sigma s_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma))\delta(\sigma - \sigma')$$
$$+ [(r_{\alpha\gamma,\sigma\delta}(\sigma, \Lambda, \Gamma) - s_{\alpha\gamma,\sigma\delta}(\sigma, \Lambda, \Gamma))\mathcal{A}_{\sigma\beta}(\sigma, \Lambda)$$
$$- \mathcal{A}_{\alpha\sigma}(\sigma, \Lambda)(r_{\sigma\gamma,\beta\delta}(\sigma, \Lambda, \Gamma) - s_{\sigma\gamma,\beta\delta}(\sigma, \Lambda, \Gamma))]\delta(\sigma - \sigma')$$
$$+ [(r_{\alpha\gamma,\beta\sigma}(\sigma, \Lambda, \Gamma) + s_{\alpha\gamma,\beta\sigma}(\sigma, \Lambda, \Gamma))\mathcal{A}_{\sigma\delta}(\Gamma, \sigma)$$
$$- \mathcal{A}_{\gamma\sigma}(\Gamma, \sigma)(r_{\alpha\sigma,\beta\delta}(\sigma, \Lambda, \Gamma) + s_{\alpha\sigma,\beta\delta}(\sigma, \Lambda, \Gamma))]\delta(\sigma - \sigma')$$
$$- 2s_{\alpha\gamma,\beta\delta}(\sigma, \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma') \tag{A.15}$$

or alternatively

$$\{J_\sigma^A(\sigma, \Lambda), J_\sigma^B(\sigma', \Gamma)\} = (r - s)^{CB}(\sigma, \Lambda, \Gamma) f_{CD}{}^A J_\sigma^D(\sigma, \Lambda)\delta(\sigma - \sigma')$$
$$+ (r + s)^{AC}(\sigma, \Lambda, \Gamma) f_{CD}{}^B J_\sigma^D(\sigma, \Gamma)\delta(\sigma - \sigma')$$
$$+ \partial_\sigma(r - s)^{AB}(\sigma, \Lambda, \Gamma)\delta(\sigma - \sigma') - 2s^{AB}(\sigma, \Lambda, \Gamma)\partial_\sigma\delta(\sigma - \sigma'). \tag{A.16}$$

Using the form of the Poisson bracket (A.15), we can calculate the algebra of monodromy matrices with distinct intervals

$$\{\mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda), \mathcal{T}_{\gamma\delta}(\xi, \xi', \Gamma)\} = \mathcal{T}_{\alpha\sigma_1}(\sigma, x_0, \Lambda)\mathcal{T}_{\gamma\sigma_2}(\xi, x_0, \Gamma)$$
$$\times [r(x_0, \Lambda, \Gamma) + \epsilon(\sigma - \xi)s(x_0, \Lambda, \Gamma)]_{\sigma_1\sigma_2,\rho_1\rho_2}\mathcal{T}_{\rho_1\beta}(x_0, \sigma', \Lambda)\mathcal{T}_{\rho_2\delta}(x_0, \xi', \Gamma)$$
$$- \mathcal{T}_{\alpha\sigma_1}(\sigma, y_0, \Lambda)\mathcal{T}_{\gamma\sigma_2}(\xi, y_0, \Gamma)[r(y_0, \Lambda, \Gamma) + \epsilon(\xi' - \sigma')s(y_0, \Lambda, \Gamma)]_{\sigma_1\sigma_2,\rho_1\rho_2}$$
$$\times \mathcal{T}_{\rho_1\beta}(y_0, \sigma', \Lambda)\mathcal{T}_{\rho_2\delta}(y_0, \xi', \Gamma), \tag{A.17}$$

where $\epsilon(x) = \text{sign}(x)$ and where we presume that σ and ξ are larger than σ' and ξ', $x_0 = \min(\sigma, \xi)$, $y_0 = \max(\sigma', \xi')$. It is important to note that in the non-ultralocal case of the algebra (A.17), due to the presence of the s-term, the function

$$\triangle^{(1)}(\sigma, \sigma', \xi, \xi', \Lambda, \Gamma) = \{\mathcal{T}_{\alpha\beta}(\sigma, \sigma', \Lambda), \mathcal{T}_{\gamma\delta}(\xi, \xi', \Gamma)\} \tag{A.18}$$

is well defined and continuous where $\sigma, \sigma', \xi, \xi'$ are all distinct, but it has discontinuities proportional to $2s$ across the hyperplanes corresponding to some of $\sigma, \sigma', \xi, \xi'$ being equal. If we want to define the Poisson bracket of transfer matrices for coinciding intervals ($\sigma = \xi, \sigma' = \xi'$) or adjacent intervals ($\sigma' = \xi \text{ or } \sigma = \xi'$), then we require the value of the discontinuous matrix-valued function $\triangle^{(1)}$ at its discontinuities. It was shown in [60] that requiring anti-symmetry of the Poisson bracket and the derivation rule to hold imposes the symmetric definition of $\triangle^{(1)}$ at its discontinuous points. For example, at $\sigma = \xi$ we must define

$$\triangle^{(1)}(\sigma, \sigma', \sigma, \xi', \Lambda, \Gamma) = \lim_{\epsilon \to 0^+} \tfrac{1}{2}(\triangle^{(1)}(\sigma, \xi, \sigma + \epsilon, \xi', \Lambda, \Gamma) + \triangle^{(1)}(\sigma, \sigma', \sigma - \epsilon, \xi', \Lambda, \Gamma))$$
$$\tag{A.19}$$

and likewise for all other possible coinciding endpoints. This definition of $\triangle^{(1)}$ at its discontinuities implies a definition of the Poisson bracket between transition matrices for coinciding and adjacent intervals that is consistent with the anti-symmetry of the Poisson

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bracket and the derivation rule. However as was shown in [59][7] this definition of the Poisson bracket $\{\mathcal{T}_{\alpha\beta}(\Lambda), \mathcal{T}_{\gamma\delta}(\Gamma)\}$ does not satisfy the Jacobi identity so that in fact no strong definition of the bracket $\{\mathcal{T}_{\alpha\beta}(\Lambda), \mathcal{T}_{\gamma\delta}(\Gamma)\}$ with coinciding or adjacent intervals can be given without violating the Jacobi identity. However, as was shown in [59] it is possible to give a weak definition of this bracket for coinciding or adjacent intervals as well. We are not going into the details of the procedure; an interesting reader can read the original paper [59] or the more recent one [61]. Let us now define

$$\Omega_{\alpha\beta}(\Lambda) = \mathcal{T}_{\alpha\beta}(\infty, -\infty, \Lambda). \tag{A.20}$$

Using the regularization procedure developed in [59], one can then show that the Poisson bracket between $\Omega_{\alpha\beta}(\Lambda)$ and $\Omega_{\gamma\delta}(\Gamma)$ takes the form

$$\{\Omega_{\alpha\beta}(\Lambda), \Omega_{\gamma\delta}(\Gamma)\} = r_{\alpha\gamma,\sigma_1\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_1\beta}(\Lambda)\Omega_{\sigma_2\delta}(\Gamma) - \Omega_{\alpha\sigma_1}(\Lambda)\Omega_{\gamma\sigma_2}(\Gamma)r_{\sigma_1\sigma_2,\beta\delta}(\Lambda, \Gamma)$$
$$+ \Omega_{\alpha\sigma_1}(\Lambda)s_{\sigma_1\gamma,\beta\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_2\delta}(\Gamma) - \Omega_{\gamma\sigma_1}(\Gamma)s_{\alpha\sigma_1,\beta\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_2\beta}(\Lambda), \tag{A.21}$$

where $r(\Lambda, \Gamma) \equiv \lim_{\sigma\to\infty} r(\Lambda, \Gamma, \sigma)$ and $s(\Lambda, \Gamma) \equiv \lim_{\sigma\to\infty} s(\Lambda, \Gamma, \sigma)$. Using (A.21), we finally obtain

$$\{\mathrm{Tr}\Omega(\Lambda), \mathrm{Tr}\Omega(\Gamma)\} = \{\Omega_{\alpha\alpha}(\Lambda), \Omega_{\gamma\gamma}(\Gamma)\} = r_{\alpha\gamma,\sigma_1\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_1\alpha}(\Lambda)\Omega_{\sigma_2\gamma}(\Gamma)$$
$$- r_{\alpha\gamma,\sigma_1\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_1\alpha}(\Lambda)\Omega_{\sigma_2\gamma}(\Gamma) + \Omega_{\alpha\sigma_1}(\Lambda)s_{\sigma_1\gamma,\alpha\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_2\gamma}(\Gamma)$$
$$- \Omega_{\alpha\sigma_1}(\Lambda)s_{\sigma_1\gamma,\alpha\sigma_2}(\Lambda, \Gamma)\Omega_{\sigma_2\gamma}(\Gamma) = 0. \tag{A.22}$$

In other words, we obtain the fact that the theory contains an infinite number of conserved charges that are in involution. This fact implies a classical integrability of given theory.

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